SOME STEADY-STATE PROBLEMS IN HEAT CONDUCTION THEORY FOR WEDGES WITH BOUNDARY CONDITIONS OF THE THIRD KIND

B. A. Vasil'ev

Inzhenerno-Fizicheskii Zhurnal, Vol. 11, No. 2, pp. 235–241, 1966

UDC 536.21

It is shown that a steady-state problem of heat conduction theory for a wedge releasing heat according to Newton's law is reduced, by means of an integral transformation, to solution of a certain functional equation. For a wedge angle of $2\gamma = \pi/m$ ($m = 1, 2, 3, \ldots$) an exact solution of the latter equation is found, and formulas for the temperature distribution are obtained.

In this paper the method of solution of heat conduction theory problems described in [1] is used to solve stationary problems for wedge-shaped bodies,

$$\Delta u = -\frac{Q}{K} \text{ in } D \quad \frac{\partial u}{\partial n} + hu|_{\Gamma} = 0, \qquad (1)$$

where Q is the amount of heat released in unit volume and unit time, K is the thermal conductivity, and h is a positive constant.

Application of a Riemann-Mellin integral transformation allows us to reduce the problem (1) for a wedge to solution of a first-order functional equation. We have examined the case of a linear source located on the wedge axis of symmetry. For wedge angles of $2\gamma = \pi/m$ ($m = 1, 2, 3 \ldots$) the solution of the functional equation is expressed in terms of the incomplete gamma function, and a complex potential of the problem is obtained. The formulas derived in the special cases $m = 1$ and $m = 2$ go over to known expressions for the potential in a half-space and a rectangular wedge. Graphs are presented for the temperature distribution along the wedge surface for the cases $m = 1, 2, 3$ and for certain values of the dimensionless parameter ha [2].

§1. **Statement of the problem, and its reduction to the functional equation.** We will examine a wedge with a vertex angle 2γ within which is located a heat source symmetrical relative to the edges ($\varphi = 0$, $\rho = a$), the source being regarded as the limiting case of a prism, and releasing a constant amount of heat per unit length. If we choose a system of cylindrical coordinates ρ, φ, z (the z axis coincides with the edge of the wedge, and the plane $\varphi = 0$ is its plane of symmetry [Fig. 1]), the problem reduces to solution of the heat conduction equation

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial T}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 T}{\partial \varphi^2} = -\frac{Q}{K},$$
$$0 < \rho < \infty, \; -\gamma < \varphi < \gamma \qquad (2)$$

with the boundary condition

$$\frac{1}{\rho}\frac{\partial T}{\partial \varphi} \pm hT\Big|_{\varphi = \pm\gamma} = 0, \quad 0 \leqslant \rho \leqslant \infty, \qquad (3)$$

where T is the unknown temperature, h is the heat transfer coefficient

$$Q = \begin{cases} \dfrac{q}{a \cdot \Delta a \cdot \Delta\varphi} & a \leqslant \rho \leqslant a + \Delta a, \\ & -\dfrac{1}{2}\Delta\varphi \leqslant \varphi \leqslant +\dfrac{1}{2}\Delta\varphi, \\ 0 \text{ elsewhere,} & \end{cases}$$

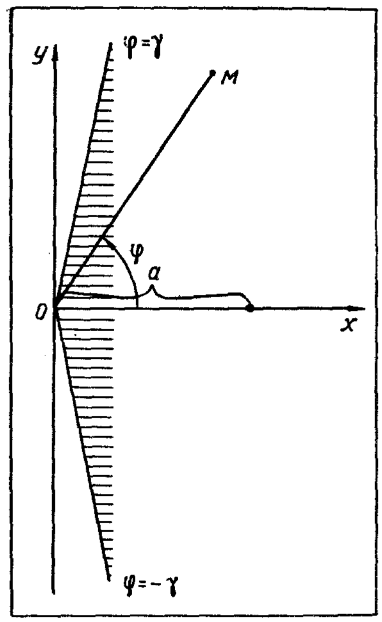

Fig. 1. Wedge with vertex angle 2γ.

and q is the amount of heat emitted by the source in unit time per unit length.

Applying a Mellin transformation to (2) and (3) and passing to the limit $\Delta a \to 0$, we obtain

$$\frac{d^2\bar{T}}{d\varphi^2} + p^2\bar{T} = \begin{cases} -\dfrac{qa^p}{K\Delta\varphi} & -\dfrac{1}{2}\Delta\varphi \leqslant \varphi \leqslant \dfrac{1}{2}\Delta\varphi \\ 0 \text{ elsewhere} & \end{cases} \qquad (4)$$

with the condition

$$\frac{d\bar{T}(p-1)}{d\varphi} \pm h\bar{T}(p)\Big|_{\varphi=\pm\gamma} = 0, \qquad (5)$$

where

$$\bar{T} = \int_0^\infty \rho^{p-1} T(\rho\varphi)\, d\rho, \quad 0 < \operatorname{Re} p < \sigma. \qquad (6)$$

The integral of (4), even with respect to φ, will be*

$$\overline{T}(p\varphi) = A(p)\cos p\varphi + \frac{qa^p \sin p(\gamma - |\varphi|)}{2Kp\cos p\gamma}. \quad (7)$$

Substituting (7) into (5), we have for A(p) the functional equation

$$hA(p+1)\cos(p+1)\gamma = pA(p)\sin p\gamma + \frac{qa^p}{2K\cos p\gamma}. \quad (8)$$

We introduce a new unknown function u(p), connected with A(p) by the relation

$$u(p) = \frac{q}{K} A(p)\cos p\gamma. \quad (9)$$

Then (8) may be written in the form

$$hu(p+1) = p \operatorname{tg} p\gamma\, u(p) + \frac{a^p}{2\cos p\gamma}. \quad (10)$$

If there is a solution of the functional equation (10), then the temperature is found from the Riemann-Mellin inversion formula

$$T = \frac{1}{2\pi i}\int_L \rho^{-p} T(p\varphi)\,dp, \quad 0 < \operatorname{Re} p < \sigma. \quad (11)$$

§2. **Solution of the functional equation.** We will examine the solution of the functional Eq. (10) under the assumption that $2\gamma = \pi/m$ ($m = 1, 2, 3, \ldots, N$) and show that in this case the solution may be expressed in finite form in terms of the incomplete gamma function.

The general solution of the functional Eq. (10) will be [5]

$$u(p) = \bar{u}(p) + Cu_0(p), \quad (12)$$

where $\bar{u}(p)$ is a particular solution of the inhomogeneous equation; $u_0(p)$ is a solution of the corresponding homogeneous equation; C is an arbitrary function with period unity.

It is not difficult to verify that

$$u_0(p) = \frac{\Gamma(p)}{h^p \prod_{k=0}^{m-1} \sin(p+k)\gamma}, \quad (13)$$

where $\gamma = \pi/2m$.

To find the particular solution of (10), we introduce the auxiliary functions

$$F(p, \alpha, \beta) = \frac{1}{2h^p}\{\exp[ha\exp(j\beta) - j\alpha]\cdot\Gamma[p;\ ha\exp(j\beta)] + \exp[ha\exp(-j\beta) + j\alpha]\cdot\Gamma[p;\ ha\exp(-j\beta)]\} \quad (14)$$

(where $\Gamma(p;z)$ is the incomplete gamma function) and

$$\Phi_m(p) = 2^{m-2}\prod_{k=1}^{m-1} \sin(p+k)\frac{\pi}{2m}, \quad m = 1, 2, 3 \ldots, \quad (15)$$

Function (14) satisfies the equation

$$hF(p+1;\ \alpha;\ \beta) = pF(p;\ \alpha;\ \beta) + a^p\cos(p\beta - \alpha) \quad (16)$$

and admits the integral representation [4]

$$F(p;\ \alpha;\ \beta) = \frac{\Gamma(p)\sin\pi p}{\pi} \times \int_0^\infty \frac{\exp(-\nu a\cos\beta)\cos(\nu a\sin\beta + \alpha)}{\nu^p(\nu + h)}\,d\nu. \quad (17)$$

Function (15) satisfies the equation

$$\Phi_m(p+1)\sin(p+1)\gamma = \Phi_m(p)\cos p\gamma, \quad (18)$$

where $\gamma = \pi/2m$, $m = 1, 2, 3, \ldots, N$.

A solution of (18) may be obtained independently of (15) in the form of a trigonometrical polynomial by using the method described in [1], which allows one to obtain the expansion

$$2^{m-2}\prod_{k=1}^{m-1}\sin(p+k)\frac{\pi}{2m} = \begin{cases} \sum_{n=0}^{s} b_n\cos\left(2np\gamma - \frac{n\pi}{2}\right) & 2\gamma = \frac{\pi}{2s+1},\ s = 0, 1, 2\ldots \\ \sum_{n=1}^{s} b_n\cos\left[(2n-1)p\gamma - \frac{(2n-1)\pi}{4}\right] & 2\gamma = \frac{\pi}{2s},\ s = 1, 2, 3\ldots, \end{cases} \quad (19)$$

where $b_n = \prod_{k=1}^{s-n}\operatorname{ctg} k\gamma$; $b_0 = \frac{1}{2}\prod_{k=1}^{s}\operatorname{ctg} k\gamma$; $\gamma = \pi/2m$.

Then the general solution of (10) may be written in the form

$$u(p) = \frac{\sum_n^s b_n F(p;\ \alpha_n;\ \beta_n) + C\Gamma(p)}{2^{m-1}\prod_{k=0}^{m-1}\sin(p+k)\gamma}, \quad (20)$$

where

$$\gamma = \pi/2m, \quad m = 1, 2, 3, \ldots, N;$$
$$b_n = \prod_{k=1}^{s-n}\operatorname{ctg} k\gamma;\ b_0 = \frac{1}{2}\prod_{k=1}^{s}\operatorname{ctg} k\gamma;$$
$$\alpha_n = n\pi/2;\ \beta_n = 2n\gamma,\ n = 0, 1, 2, \ldots, s,$$
$$m = 2s+1, \quad s = 0, 1, 2\ldots;$$
$$\alpha_n = (2n-1)\pi/4,\ \beta_n = (2n-1)\gamma,\ n = 1, 2, 3, \ldots, s,$$
$$m = 2s,\ s = 1, 2, 3\ldots.$$

*It is necessary to go to the limit $\Delta\varphi \to 0$.

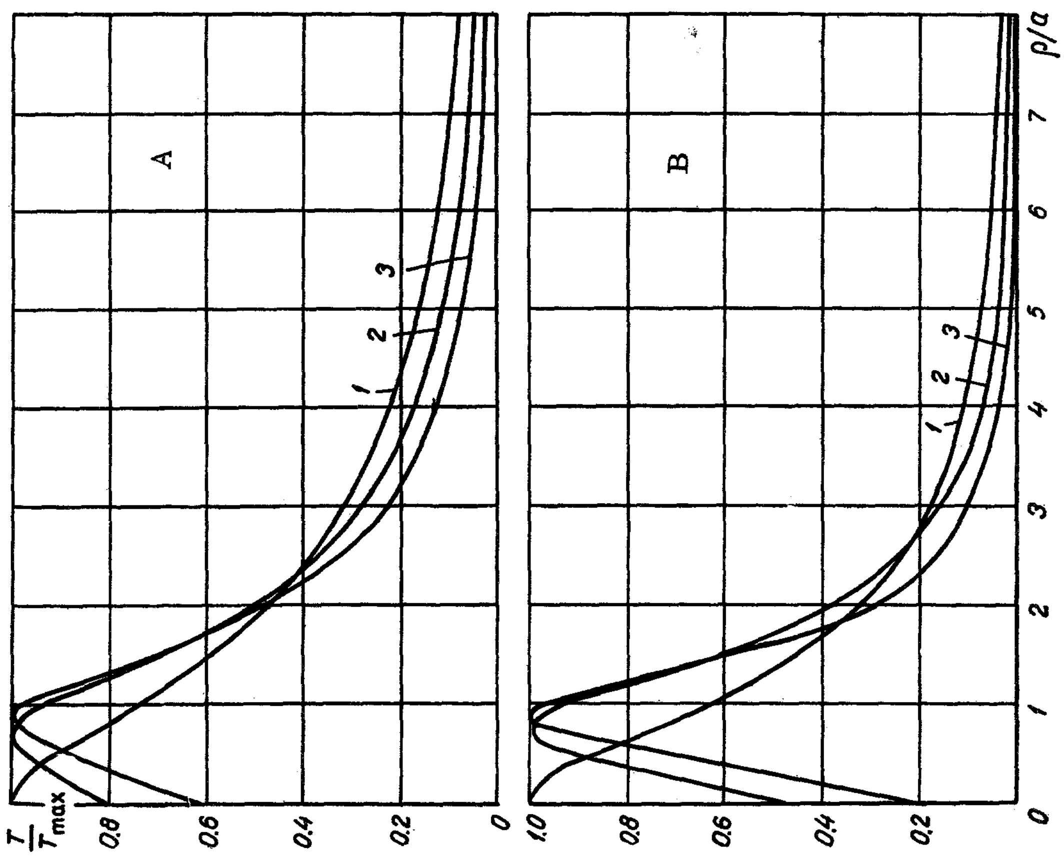

Fig. 3. Dependence of the temperature on ρ/a with $ha = 0.5$ (A) and 2 (B) and $\varphi = \pm\gamma$; 1, 2, 3) with m = 1, 2, and 3, respectively.

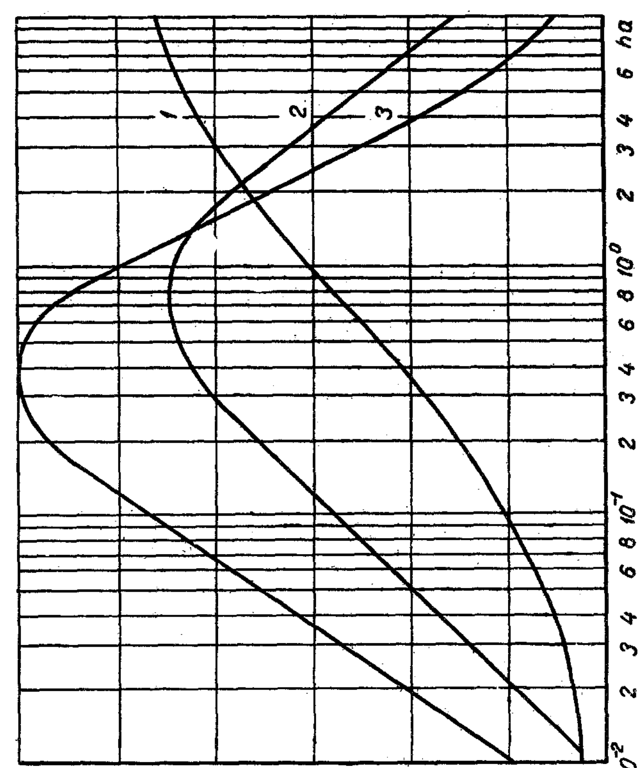

Fig. 2. Dependence of $q_0/q_m = f(ha)$: 1, 2, and 3—for m = 1, 2, and 3, respectively.

Formula (20) is a generalization of the well-known cases of a half-plane (m = 1) and a rectangular wedge (m = 2). In these cases C = 0.*

§3. **Temperature distribution in a wedge with vertex angle $2\gamma = \pi/m$ (m = 1, 2, 3, ..., N).** The formal solution of the problem examined may be obtained with the aid of the Riemann-Mellin inversion formula

$$T = \frac{q}{4\pi iK}\left\{\int_L \left(\frac{a}{\rho}\right)^p \frac{\sin(\gamma - |\varphi|)}{p\cos p\gamma}\,dp + \right.$$

$$\left. + \int_L \rho^{-p} u(p) \frac{\cos p\varphi}{\cos p\gamma}\,dp\right\} =$$

$$= T_1 + T_2. \qquad (21)$$

Evaluation of the first integral gives [3]

$$T_1 = \frac{q}{4\pi K} \ln \frac{\rho^{2m} + 2a^m\rho^m\cos m\varphi + a^{2m}}{\rho^{2m} - 2a^m\rho^m\cos m\varphi + a^{2m}}. \qquad (22)$$

We will briefly examine one possible method of calculating the second contour integral, for which the following equalities are required [4]:

$$\sin\pi p = 2^{2m-1}\prod_{k=0}^{m-1}\sin(p+k)\frac{\pi}{2m}\prod_{k=0}^{m-1}\cos(p+k)\frac{\pi}{2m},$$

$$m = 1, 2, 3, \ldots, N; \qquad (23)$$

$$\frac{1}{2\pi i}\int_L \frac{\Gamma(p)\cos(p\beta + \alpha)}{(\nu\rho)^p}\,dp =$$

$$= \exp(-\nu\rho\cos\beta)\cos(\nu\rho\sin\beta + \alpha), \qquad (24)$$

where $\operatorname{Re} p > 0$; $|\beta| < \pi/2$.

From (17), (19), (20), and (23) we obtain

$$u(p)\frac{\cos p\varphi}{\cos p\gamma} =$$

$$= \frac{2\Gamma(p)}{\pi}\int_0^\infty \left\{\sum_n^s \sum_k^s b_n b_k \exp(-\nu a\cos\beta_n) \times \right. \qquad (25)$$

$$\left. \times \cos(\nu a\sin\beta_n + \alpha_n)\cdot\cos[(\beta_k \pm \varphi)p + \alpha_k]/\nu^p(\nu + h)\right\} d\nu.$$

Substituting (25) into (21) and changing the order of integration, we find from (24) that

$$T_2 = \frac{2q}{\pi K}\int_0^\infty \left\{\sum_n^s \sum_k^s b_n b_k \exp\{-\nu a\cos\beta_n - \right.$$

$$-\nu\rho\cos(\beta_k \pm \varphi)\}\times$$

$$\times\cos(\nu a\sin\beta_n + \alpha_n)\times$$

$$\left. \times\cos[\nu\rho\sin(\beta_k \pm \varphi) + \alpha_k]/(\nu + h)\right\} d\nu. \qquad (26)$$

The integral in (26) may be calculated in the complex plane $\xi = \rho\exp(j\varphi)$ and expressed in terms of the integral exponential function $E_1(z)$ [6]. Following simple transformations the complex potential may be written in the form*

$$W = \frac{q}{2\pi K}\ln\frac{\xi^m + a^m}{\xi^m - a^m} +$$

$$+ \frac{q}{\pi K}\sum_n^s \sum_k^s b_n b_k \exp[\pm j\alpha_n \pm j\alpha_k + ha\exp(\mp j\beta_n) +$$

$$+ h\xi\exp(\mp j\beta_k)] \times E_1[ha\exp(\mp j\beta_n) + h\xi\exp(\mp j\beta_k)], \qquad (27)$$

where

$$b_n = \prod_{k=1}^{s-n}\operatorname{ctg} k\gamma, \quad b_0 =$$

$$= \frac{1}{2}\prod_{k=1}^{s}\operatorname{ctg} k\gamma, \quad \gamma = \pi/2m, \quad m = 1, 2, 3, \ldots, N;$$

$$\beta_n = 2n\gamma, \quad \alpha_n = n\pi/2, \quad n = 0,$$

$$1, 2, \ldots, s, \quad m = 2s + 1, \quad s = 0, 1, 2 \ldots;$$

$$\beta_n = (2n-1)\gamma, \quad \alpha_n = (2n-1)\pi/4,$$

$$n = 1, 2, 3, \ldots, s, \quad m = 2s,$$

$$s = 1, 2, 3 \ldots;$$

$$E_1(z) = \int_z^\infty \frac{\exp(-u)}{u}\,du.$$

Figure 2 shows the dependence of heat flux density at the wedge tip ($\varphi = \gamma$; $\rho \to 0$) on the parameter ha. The limiting values $ha \to \infty$ and $ha \to 0$ of the parameter correspond to the well-known cases of boundary conditions of the first and second kinds. Figure 3 shows the derived curves of temperature distribution for the cases m = 1, 2, 3 and for two values of the parameter ha. The maximum value of temperature is reached at approximately the points of projection of the source on the wedge edge.

SUMMARY

The method of integral transforms with subsequent reduction of the problem to solution of the functional equations permits us the effective solution of certain problems in heat conduction theory with a boundary condition of the third kind [1]. Using this method, it is not difficult to generalize the results obtained above and to construct the Green's function for a plane stationary problem in potential theory for wedge-shaped regions with a boundary condition of the third kind.

*It may be shown that C = 0 for any m.

*We have in mind the following combination of signs:

$$\begin{pmatrix}+j\alpha_n; & -j\beta_n\\ +j\alpha_k; & -j\beta_k\end{pmatrix}; \begin{pmatrix}-j\alpha_n; & +j\beta_n\\ +j\alpha_k; & -j\beta_k\end{pmatrix};$$

$$\begin{pmatrix}+j\alpha_n; & -j\beta_n\\ -j\alpha_k; & +j\beta_k\end{pmatrix}; \begin{pmatrix}-j\alpha_n; & +j\beta_n\\ -j\alpha_k; & +j\beta_k\end{pmatrix}.$$

REFERENCES

1. N. N. Lebedev and I. P. Skal'skaya, ZhTF, **34**, 1556, 1964.
2. H. Carslaw and J. Jaeger, Conduction of Heat in Solids [Russian translation], Izd. Nauka, 1964.
3. N. N. Lebedev, I. P. Skal'skaya, and Ya. S. Uflyand, Collected Problems in Mathematical Physics [in Russian], GITTL, 1955.
4. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series and Products [in Russian], FM, 1962.
5. Milne-Thomson, The Calculus of Finite Differences, Macmillan, London, 1951.
6. L. S. Bark, Tables of an Integral Exponential Function in the Complex Domain [in Russian], VTs. AN SSSR, no. 31, 1965.

11 February 1966 Leningrad Polytechnic Institute